

# ON THE CLASSIFICATION OF FINITE DIMENSIONAL IRREDUCIBLE MODULES FOR AFFINE BMW ALGEBRAS

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ABSTRACT. In this paper, we classify the finite dimensional irreducible modules for affine BMW algebra over an algebraically closed field with arbitrary characteristic.

## 1. INTRODUCTION

In [19], Haering-Oldenburg introduced a class of associative algebras called affine Birman-Murakami-Wenzl (BMW for brevity) algebras in order to study knot invariants. These algebras, which can be considered as the affinization of BMW algebras in [6, 21], had been studied extensively by many authors in [9, 11, 12, 14, 15, 18, 23, 25–29, 32–34] etc.

Recently, Goodman [14] studied the cyclotomic quotient of affine BMW algebras in  $d$ -semi-admissible case (see Definition 2.13 for details). This sets up the relationship between the representations of cyclotomic BMW algebras in general case and those for the cyclotomic BMW algebras in  $\mathbf{u}$ -admissible case in [28, 29]. Using the results on the classification of irreducible modules of cyclotomic BMW algebras in [28, 29, 35], we get all finite dimensional irreducible modules for affine BMW algebras over an algebraically field  $\kappa$  with arbitrary characteristic.

In order to classify the finite dimensional irreducible modules for affine BMW algebras over  $\kappa$ , we have to determine whether two irreducible modules for different cyclotomic BMW algebras are isomorphic as the modules for the affine BMW algebra. For this, we need the result on the classification of finite dimensional irreducible modules for extended affine Hecke algebra  $\hat{\mathcal{H}}_n$  of type  $A_{n-1}$  as follows.

The first result on the classification of irreducible  $\hat{\mathcal{H}}_n$ -modules is due to Bernstein and Zelevinsky [7, 37], who classified the irreducible  $\hat{\mathcal{H}}_n$ -modules over  $\mathbb{C}$  when the defining parameter  $q$  is not a root of unity. In this case, they used multisegments of length  $n$  to index the complete set of non-isomorphic irreducible  $\hat{\mathcal{H}}_n$ -modules. In [24], Rogawski gave a different method to reprove Bernstein and Zelevinsky's result. Note that Kazhdan-Lusztig [20] and Xi [36] classified the finite dimensional

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irreducible modules for affine Hecke algebras in any type. In particular, their results contain the case for extended affine Hecke algebras of type  $A_{n-1}$ .

On the other hand, any irreducible  $\hat{\mathcal{H}}_n$ -module over  $\kappa$  can be realized as an irreducible module for an Ariki-Koike algebra [2]. In the later case, its irreducible modules are indexed by Kleshchev multipartitions [1]. In [31], Vazirani gave the explicit relationship between the set of Kleshchev multi-partitions and the set of multi-segments when  $q$  is not a root of unity. If  $q$  is a root of unity, the irreducible  $\hat{\mathcal{H}}_n$ -modules have been classified via aperiodic multisegments in [16] (resp. [4]) over  $\mathbb{C}$  (resp. over  $\kappa$ ). Further, Ariki-Jacon-Lecouvey set up the explicit relationship between the set of Kleshchev multipartitions and the set of aperiodic multi-segments in [3, Theorem 6.2] over  $\kappa$ . This is the result that we need when we classify the finite dimensional irreducible modules for affine BMW algebras over  $\kappa$ .

Throughout, let  $\kappa$  be an algebraically closed field which contains non-zero elements  $q, \varrho, \delta$  and a family of elements  $\Omega = \{\omega_i \mid i \in \mathbb{Z}\}$  such that  $\delta = q - q^{-1}$  and  $\omega_0 = 1 - \delta^{-1}(\varrho - \varrho^{-1})$ . Let  $n$  be a positive integer with  $n \geq 2$ .

**Definition 1.1.** [19] The affine BMW algebra  $\hat{\mathcal{B}}_n$  is the unital associative  $\kappa$ -algebra generated by  $g_i, e_i, x_1^{\pm 1}$ ,  $1 \leq i \leq n-1$  subject to the following relations:

- (1)  $x_1 x_1^{-1} = x_1^{-1} x_1 = 1$  and  $g_i g_i^{-1} = g_i^{-1} g_i = 1$ , for  $1 \leq i \leq n$ ,
- (2)  $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$ , for  $1 \leq i < n-1$ ,
- (3)  $g_i g_j = g_j g_i$  if  $|i-j| > 1$ ,
- (4)  $x_1 g_1 x_1 g_1 = g_1 x_1 g_1 x_1$ , and  $x_1 g_j = g_j x_1$  for  $j \geq 2$ ,
- (5)  $e_i^2 = \omega_0 e_i$ , for  $1 \leq i < n$ ,
- (6)  $e_1 x_1^a e_1 = \omega_a e_1$ , for  $a \in \mathbb{Z}^{>0}$ ,
- (7)  $x_1 g_j = g_j x_1$ , for  $2 \leq j \leq n-1$ ,
- (8)  $g_i e_j = e_j g_i$ , and  $e_i e_j = e_j e_i$  if  $|i-j| > 1$ ,
- (9)  $e_i g_i = \varrho g_i = g_i e_i$ , for  $1 \leq i \leq n-1$ ,
- (10)  $e_i g_{i\pm 1} e_i = \varrho e_i$ ,  $e_i e_{i\pm 1} e_i = e_i$ ,
- (11)  $g_i g_{i\pm 1} e_i = e_{i\pm 1} e_i$  and  $e_i g_{i\pm 1} g_i = e_i e_{i\pm 1}$ ,
- (12)  $g_i - g_i^{-1} = \delta(1 - e_i)$ , for  $1 \leq i < n$ ,
- (13)  $e_1 x_1 g_1 x_1 g_1 = e_1 = g_1 y_1 g_1 x_1 e_1$ .

By Definition 1.1, there is an anti-involution  $*$ :  $\hat{\mathcal{B}}_n \rightarrow \hat{\mathcal{B}}_n$  which fixes  $g_i, e_i$  and  $x_1$ ,  $1 \leq i \leq n-1$ . Further, it is pointed in [18, (2.1)] that Turaev [30] has proved that  $e_1 x_1^{-a} e_1 = \omega_{-a} e_1$  for  $a \in \mathbb{Z}^{>0}$  and  $\omega_{-a}$  is a polynomial in  $\omega_b$  for  $b \in \mathbb{Z}^{>0}$ . Therefore,  $\omega_a$  is well-defined for all  $a \in \mathbb{Z}$ .

Goodman and Hauschild-Mosley [18] constructed a basis for  $\hat{\mathcal{B}}_n$  and showed that  $\hat{\mathcal{B}}_n$  is of infinite dimension. In fact, Goodman and Hauschild-Mosley's results [18] are available over an integral domain.

It is well-known that an affine Wenzl algebra in [22] can be considered as a degenerate affine BMW algebra. Ariki, Mathas and Rui [5] constructed an infinite dimensional irreducible modules for affine Wenzl algebra. Mimicking this construction, we know that  $\hat{\mathcal{B}}_n$  has infinite dimensional irreducible modules over a field. In other words,  $\hat{\mathcal{B}}_n$  is not finitely generated over its center. For the description of the center of  $\hat{\mathcal{B}}_n$ , see [9].

The aim of this paper is to classify all finite dimensional irreducible  $\hat{\mathcal{B}}_n$ -modules over  $\kappa$ . Before we state our main result, we need the notion of aperiodic multi-segments in [4].

Let  $e$  be the smallest positive integer such that

$$1 + q^2 + q^4 + \cdots + q^{2(e-1)} = 0$$

in  $\kappa^1$ . If there is no such positive integer, then we set  $e = \infty$ . In other words,  $e$  is the order of  $q^2 \in \kappa$ . Recall that a segment  $\Delta$  of length  $j = |\Delta|$  is a sequence of consecutive residues  $[i, i+1, \dots, i+j-1]$  where  $i, i+1, \dots, i+j-1 \in \mathbb{Z}_e$ . An multi-segment  $\Delta$  is an unordered collection of segments  $\Delta_i$  with length  $\sum_i |\Delta_i|$ . Following [4], we says that  $\Delta$  is aperiodic if for every  $j$ , there is an  $i \in \mathbb{Z}_e$  such that  $[i, i+1, \dots, i+j-1]$  does not appear in  $\Delta$ . Let  $\mathcal{M}_e^n$  be the set of all aperiodic multi-segments with length  $n$ . The following is the main result of this paper, which gives the classification of finite dimensional irreducible  $\hat{\mathcal{B}}_n$ -modules over  $\kappa$ .

**Theorem 1.2.** *Let  $\hat{\mathcal{B}}_n$  be the affine BMW algebra over  $\kappa$ .*

- a) *Any finite dimensional irreducible  $\hat{\mathcal{B}}_n$ -modules is of form  $D^{f,\lambda}$  where  $D^{f,\lambda}$ , defined via the cellular basis of some cyclotomic quotient  $\mathcal{B}_{r,n}(\mathbf{u})$  of  $\hat{\mathcal{B}}_n$  in Theorem 2.9, is an irreducible  $\mathcal{B}_{r,n}(\mathbf{u})$ -module such that*
  - (i)  *$0 \leq f \leq \lfloor n/2 \rfloor$  and  $\lambda$  is a Kleshchev multipartition of  $n - 2f$  in the sense of [4]. Further, if  $\omega_a = 0$  for all  $a \in \mathbb{Z}$  and if  $2 \mid n$ , then  $f \neq n/2$ .*
  - (ii)  *$\mathbf{u}$ -admissible condition holds for  $\mathcal{B}_{r,n}(\mathbf{u})$  if  $f > 0$ .*
- b) *Let  $D^{f,\lambda}$  (resp.  $D^{\ell,\mu}$ ) be the irreducible  $\mathcal{B}_{r,n}(\mathbf{u})$  (resp.  $\mathcal{B}_{s,n}(\mathbf{v})$ )-module. Then  $D^{f,\lambda} \cong D^{\ell,\mu}$  as  $\hat{\mathcal{B}}_n$ -modules if and only if  $f = \ell$  and the images of  $\lambda$  and  $\mu$  under the map of [3, Theorem 6.2] are the same aperiodic multisegment in  $\mathcal{M}_e^{n-2f}$ .*

We remark that each aperiodic multisegment of length  $n$  indexes an irreducible  $\hat{\mathcal{B}}_n$ -module on which  $e_1$  acts trivially. This follows from Ariki-Mathas's result on the classification of irreducible  $\mathcal{H}_n$ -modules in [4]. However, we can not say that any pair  $(f, \Delta)$  with  $0 < f < \lfloor n/2 \rfloor$  and  $\Delta \in \mathcal{M}_e^{n-2f}$  indexes an irreducible  $\hat{\mathcal{B}}_n$ -module. The reason is that each  $\Delta \in \mathcal{M}_e^{n-2f}$  corresponds at least a Kleshchev

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<sup>1</sup>The current  $q^2$  is  $q$  in [4].

multi-partition with respect to a family of parameters  $u_1, u_2, \dots, u_r \in \kappa^*$ . However, we do not know whether the  $\mathbf{u}$ -admissible condition holds for  $\mathcal{B}_{r,n}(\mathbf{u})$ .

The content of this paper is organized as follows. We recall some of results on the representations of  $\mathcal{B}_{r,n}(\mathbf{u})$  in section 2 and prove Theorem 1.2 in section 3.

## 2. CYCLOTOMIC BMW ALGEBRAS

In this section, we recall some results on the cyclotomic BMW algebra over  $\kappa$  although some of them hold over an integral domain. Throughout, we assume  $r \in \mathbb{Z}$  with  $r \geq 1$ .

**Definition 2.1.** [19] Let  $I$  be the two-sided ideal of  $\hat{\mathcal{B}}_n$  generated by the cyclotomic polynomial

$$(2.2) \quad f(x_1) = (x_1 - u_1)(x_1 - u_2) \cdots (x_1 - u_r),$$

where  $u_i \in \kappa^*$ ,  $1 \leq i \leq r$ . The cyclotomic BMW algebra  $\mathcal{B}_{r,n}(\mathbf{u})$  is the quotient algebra  $\hat{\mathcal{B}}_n/I^2$ .

**Remark 2.3.** When  $r = 1$ ,  $\mathcal{B}_{r,n}(\mathbf{u})$  is the usual BMW algebra, which was introduced by Birman-Wenzl [6] and independently by Murakami [21].

It is known that  $\mathcal{B}_{r,n}(\mathbf{u})$  can be used to study the finite dimensional irreducible  $\hat{\mathcal{B}}_n$ -modules over  $\kappa$ . Pick a finite dimensional irreducible  $\hat{\mathcal{B}}_n$ -module  $M$  over  $\kappa$ . Let  $f(x_1)$  be the characteristic polynomial of  $x_1$  with respect to  $M$ . Then  $M$  has to be an irreducible  $\mathcal{B}_{r,n}(\mathbf{u})$ -module where  $\mathcal{B}_{r,n}(\mathbf{u}) = \hat{\mathcal{B}}_n/I$  and  $I$  is the two-sided ideal of  $\hat{\mathcal{B}}_n$  generated by  $f(x_1)$ . Since  $\kappa$  is an algebraically close field,  $f(x_1)$  is given in (2.2) for some  $u_1, u_2, \dots, u_r \in \kappa$ . Further,  $u_i \in \kappa^*$  since  $x_1$  is invertible in  $\mathcal{B}_{r,n}(\mathbf{u})$ . Therefore, we will get all finite dimensional irreducible  $\hat{\mathcal{B}}_n$ -modules over  $\kappa$  if we classify the irreducible  $\mathcal{B}_{r,n}(\mathbf{u})$ -modules for all  $\mathbf{u} \in (\kappa^*)^r$  and  $r \geq 1$ .

**Definition 2.4.** [14] We say that the  $d$ -semi-admissible condition holds for  $\mathcal{B}_{r,n}(\mathbf{u})$  if  $d$  is the minimal integer such that  $\{e_1, e_1 x_1, \dots, e_1 x_1^d\}$  is linear dependent in  $\mathcal{B}_{r,2}(\mathbf{u})$ .

Obviously,  $0 \leq d \leq r$ . We have  $e_1 = 0$  if  $d = 0$ . In this case, there is no restriction on  $u_i$ 's. Further,  $\mathcal{B}_{r,n}(\mathbf{u})$  is the Ariki-Koike algebra  $\mathcal{H}_{r,n}$  [2] whose simple modules have been classified in [1].

If  $d = r$ , then the  $d$ -semi-admissible condition is the  $\mathbf{u}$ -admissible condition in [29] or admissible conditions in [33]. In particular,  $\mathbf{u}$ -admissible condition always holds if  $e_1 \neq 0$  and  $r = 1$ .

In  $\mathbf{u}$ -admissible case, we have [29]

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<sup>2</sup>In [19], Haering-Oldenburg defined  $\mathcal{B}_{r,n}(\mathbf{u})$  without assuming  $u_i \in \kappa^*$ ,  $1 \leq i \leq r$ .

$$(2.5) \quad \varrho^{-1} = \alpha \prod_{\ell=1}^r u_{\ell}, \text{ and } \omega_a = \sum_{j=1}^r u_j^a \gamma_j, \forall a \in \mathbb{Z},$$

where

- (1)  $\gamma_i = (\gamma_r(u_i) + \delta^{-1} \varrho(u_i^2 - 1) \prod_{j \neq i} u_j) \prod_{j \neq i} \frac{u_i u_j - 1}{u_i - u_j}$ , and  $\gamma_r(z)$  is 1 (resp.  $-z$ ) if  $2 \nmid r$  (resp. otherwise).
- (2)  $\alpha \in \{1, -1\}$  if  $2 \nmid r$  and  $\alpha \in \{q^{-1}, -q\}$ , otherwise.
- (3)  $\omega_0 = \delta^{-1} \varrho(\prod_{\ell=1}^r u_{\ell}^2 - 1) + 1 - \frac{(-1)^r + 1}{2} \alpha^{-1} \varrho^{-1}$ .

We have the following result, which will be used when we prove Theorem 1.2.

**Lemma 2.6.** *Suppose  $\mathbf{u}$ -admissible condition holds for  $\mathcal{B}_{r,2}(\mathbf{u})$ . We have  $\omega_i \neq 0$  for some  $i, 0 \leq i \leq r-1$  if there is a  $j \in \mathbb{Z}$  such that  $\omega_j \neq 0$ .*

*Proof.* This follows from Definitions 2.1 and 1.1(6).  $\square$

If the  $\mathbf{u}$ -admissible condition holds, then  $\mathcal{B}_{r,n}(\mathbf{u})$  is (weakly) cellular in the sense of [17] as follows.

**Definition 2.7.** [17] Assume that  $R$  is a commutative ring with the multiplicative identity 1. Let  $A$  be an  $R$ -algebra. Fix a partially ordered set  $\Lambda = (\Lambda, \triangleright)$  and for each  $\lambda \in \Lambda$  let  $T(\lambda)$  be a finite set. Finally, fix  $\mathbf{m}_{\mathbf{s}\mathbf{t}} \in A$  for all  $\lambda \in \Lambda$  and  $\mathbf{s}, \mathbf{t} \in T(\lambda)$ . Then the triple  $(\Lambda, T, C)$  is a **cell datum** for  $A$  if:

- a)  $\mathcal{M} = \{ \mathbf{m}_{\mathbf{s}\mathbf{t}} \mid \lambda \in \Lambda \text{ and } \mathbf{s}, \mathbf{t} \in T(\lambda) \}$  is an  $R$ -basis for  $A$ ;
- b) the  $R$ -linear map  $*$  :  $A \rightarrow A$  determined by  $(\mathbf{m}_{\mathbf{s}\mathbf{t}})^* = \mathbf{m}_{\mathbf{t}\mathbf{s}}$ , for all  $\lambda \in \Lambda$  and all  $\mathbf{s}, \mathbf{t} \in T(\lambda)$  is an anti-isomorphism of  $A$ ;
- c) for all  $\lambda \in \Lambda$ ,  $\mathbf{s} \in T(\lambda)$  and  $a \in A$  there exist scalars  $r_{\mathbf{t}\mathbf{u}}(a) \in R$  such that

$$\mathbf{m}_{\mathbf{s}\mathbf{t}} a = \sum_{\mathbf{u} \in T(\lambda)} r_{\mathbf{t}\mathbf{u}}(a) \mathbf{m}_{\mathbf{s}\mathbf{u}} \pmod{A^{\triangleright \lambda}},$$

where  $A^{\triangleright \lambda} = R\text{-span} \{ \mathbf{m}_{\mathbf{u}\mathbf{v}} \mid \mu \triangleright \lambda \text{ and } \mathbf{u}, \mathbf{v} \in T(\mu) \}$ . Furthermore, each scalar  $r_{\mathbf{t}\mathbf{u}}(a)$  is independent of  $\mathbf{s}$ .

An algebra  $A$  is a **cellular algebra** if it has a cell datum and in this case we call  $\mathcal{M}$  a cellular basis of  $A$ .

The notion of weakly cellular algebras in [13] is obtained from Definition 2.7 by using

$$(\mathbf{m}_{\mathbf{s}\mathbf{t}})^* \equiv \mathbf{m}_{\mathbf{t}\mathbf{s}} \pmod{A^{\triangleright \lambda}}$$

instead of  $(\mathbf{m}_{\mathbf{s}\mathbf{t}})^* = \mathbf{m}_{\mathbf{t}\mathbf{s}}$ . Note that both cellular algebras and weakly cellular algebras are standardly based algebras in the sense of [10]. From this, one can see that cellular algebras and weakly cellular algebras share the similar results on

representation theory. For this reason, both cellular algebras and weakly cellular algebras will be called cellular algebras later on.

Now, we briefly recall the representation theory of cellular algebras over a field in [17]. We remark that all modules considered in this paper are right modules.

Every irreducible  $A$ -module arises in a unique way as the simple head of some cell module. For each  $\lambda \in \Lambda$  fix  $\mathfrak{s} \in T(\lambda)$  and let

$$\mathfrak{m}_{\mathfrak{t}} = \mathfrak{m}_{\mathfrak{s}\mathfrak{t}} + A^{\triangleright\lambda}.$$

The cell module  $S^\lambda$  of  $A$  with respect to  $\lambda$  can be considered as the free  $R$ -module with basis  $\{\mathfrak{m}_{\mathfrak{t}} \mid \mathfrak{t} \in T(\lambda)\}$ . The cell module  $S^\lambda$  comes equipped with a natural symmetric bilinear form  $\phi_\lambda$  which is determined by the equation

$$\mathfrak{m}_{\mathfrak{s}\mathfrak{t}}\mathfrak{m}_{\mathfrak{t}'\mathfrak{s}} \equiv \phi_\lambda(\mathfrak{m}_{\mathfrak{t}}, \mathfrak{m}_{\mathfrak{t}'} \cdot \mathfrak{m}_{\mathfrak{s}\mathfrak{s}}) \pmod{A^{\triangleright\lambda}}.$$

The bilinear form  $\phi_\lambda$  is  $A$ -invariant in the sense that

$$\phi_\lambda(xa, y) = \phi_\lambda(x, ya^*), \text{ for } x, y \in S^\lambda \text{ and } a \in A.$$

Consequently,

$$\text{Rad } S^\lambda = \{x \in S^\lambda \mid \phi_\lambda(x, y) = 0 \text{ for all } y \in S^\lambda\}$$

is an  $A$ -submodule of  $S^\lambda$  and  $D^\lambda = S^\lambda / \text{Rad } S^\lambda$  is either zero or absolutely irreducible.

Graham and Lehrer [17] have proved that all non-zero  $D^\lambda$  consist of a complete set of pairwise non-isomorphic irreducible  $A$ -modules. This gives a useful method to classify the irreducible modules for cellular algebras.

Recall that a **composition**  $\lambda = (\lambda_1, \lambda_2, \dots)$  of  $m$  is a sequence of non-negative integers with  $|\lambda| = \sum_i \lambda_i = m$ . If  $\lambda$  is weakly decreasing, then  $\lambda$  is called a partition. Similarly, an  $r$ -partition of  $m$  is an ordered  $r$ -tuple  $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)})$  of partitions  $\lambda^{(s)}$  with  $1 \leq s \leq r$  such that  $|\lambda| = \sum_{i=1}^r |\lambda^{(i)}| = m$ . Let  $\Lambda_r^+(n)$  be the set of all  $r$ -partitions of  $n$ . We say that  $\mu$  dominates  $\lambda$  and write  $\lambda \trianglelefteq \mu$  if

$$\sum_{j=1}^{i-1} |\lambda^{(j)}| + \sum_{k=1}^l \lambda_k^{(i)} \leq \sum_{j=1}^{i-1} |\mu^{(j)}| + \sum_{k=1}^l \mu_k^{(i)}$$

for  $1 \leq i \leq r$  and  $l \geq 0$ . So,  $(\Lambda_r^+(n), \trianglelefteq)$  is a poset. If  $\lambda \trianglelefteq \mu$  and  $\lambda \neq \mu$ , we write  $\lambda \triangleleft \mu$ . Let

$$(2.8) \quad \Lambda_{r,n} = \{(k, \lambda) \mid 0 \leq k \leq \lfloor n/2 \rfloor, \lambda \in \Lambda_r^+(n - 2k)\}.$$

Then  $\Lambda_{r,n}$  is a poset with  $\trianglerighteq$  as the partial order on it. More explicitly,  $(k, \lambda) \trianglerighteq (\ell, \mu)$  for  $(k, \lambda), (\ell, \mu) \in \Lambda_{r,n}$  if either  $k > \ell$  in the usual sense or  $k = \ell$  and  $\lambda \trianglerighteq \mu$ . Here  $\trianglerighteq$  is the dominance order defined on  $\Lambda_r^+(n - 2k)$ .

The following theorem is well-known. See [13, 34] for another description of cellular basis for  $\mathcal{B}_{r,n}(\mathbf{u})$ .

**Theorem 2.9.** [29] *Suppose that the  $\mathbf{u}$ -admissible condition holds for  $\mathcal{B}_{r,n}(\mathbf{u})$ . Then*

$$\mathcal{C} = \bigcup_{(f,\lambda) \in \Lambda_{r,n}} \{C_{\mathbf{st}} \mid \mathbf{s}, \mathbf{t} \in T(f, \lambda)\}$$

*is a weakly cellular basis of  $\mathcal{B}_{r,n}(\mathbf{u})$  over the poset  $\Lambda_{r,n}$ . In this case, the required  $\kappa$ -linear anti-involution on  $\mathcal{B}_{r,n}(\mathbf{u})$  is  $*$ , which fixes  $g_i, e_i$  and  $x_1, 1 \leq i \leq n-1$ . In particular, the rank of  $\mathcal{B}_{r,n}(\mathbf{u})$  is  $r^n(2n-1)!!$ .*

In this paper, we do not need the explicit definition of  $C_{\mathbf{st}}$  in [29, 4.17]. What we will need is some properties of cell modules  $S^{f,\lambda}$ ,  $(f, \lambda) \in \Lambda_{r,n}$  for  $\mathcal{B}_{r,n}(\mathbf{u})$  with respect to the cellular basis in Theorem 2.9. Let  $\phi_{f,\lambda}$  be the invariant form on the cell module  $S^{f,\lambda}$  with respect to  $\lambda \in \Lambda_{r,n}$ .

We have  $\mathcal{B}_{r,n}(\mathbf{u})/I \cong \mathcal{H}_{r,n}(\mathbf{u})$ , where  $\mathcal{H}_{r,n}(\mathbf{u})$  is the Ariki-Koike algebra [2] and  $I$  is the two-sided ideal of  $\mathcal{B}_{r,n}(\mathbf{u})$  generated by the cyclotomic polynomial  $f(x_1)$  in (2.2). The image of the cellular basis of  $\mathcal{B}_{r,n}(\mathbf{u})$  in Theorem 2.9 is the cellular basis of  $\mathcal{H}_{r,n}(\mathbf{u})$  in [8]. The corresponding cell module of  $\mathcal{H}_{r,n}(\mathbf{u})$  with respect to  $\lambda \in \Lambda_r^+(n)$  is denoted by  $S^\lambda$ . Let  $\phi_\lambda$  be the invariant form on  $S^\lambda$ .

**Proposition 2.10.** [29, 5.2] *Suppose that the  $\mathbf{u}$ -admissible condition holds for  $\mathcal{B}_{r,n}(\mathbf{u})$  over  $\kappa^3$ . Assume that  $(f, \lambda) \in \Lambda_{r,n}$ .*

- a) *If  $f \neq n/2$ , then  $\phi_{f,\lambda} \neq 0$  if and only if  $\phi_\lambda \neq 0$ .*
- b) *If  $\omega_a \neq 0$  for some non-negative integer  $a \leq r-1$ , then  $\phi_{n/2,0} \neq 0$ .*
- c) *If  $\omega_a = 0$  for all non-negative integers  $a \leq r-1$ , then  $\phi_{n/2,0} = 0$ .*

Note that  $\phi_\lambda \neq 0$  if and only if  $D^\lambda \neq 0$  for  $\mathcal{H}_{r,n-2f}(\mathbf{u})$ . By [1],  $\phi_\lambda \neq 0$  if and only if  $\lambda$  is a Kleshchev multipartition in the sense of the Definition in [4, p605]. So, the irreducible  $\mathcal{B}_{r,n}(\mathbf{u})$ -modules are classified via Proposition 2.10. More explicitly, we have the following result which can be found in [35] for  $r=1$  and [29] for  $r \geq 2$ . We remark that the  $\mathbf{u}$  admissible condition always holds for  $r=1$  and  $e_1 \neq 0$ .

**Theorem 2.11.** [29, 35] *Suppose that  $\mathbf{u}$ -admissible condition holds for  $\mathcal{B}_{r,n}(\mathbf{u})$  over  $\kappa$ .*

- a) *If either  $\omega_a \neq 0$  for some non-negative integer  $a \leq r-1$  or  $\omega_a = 0$  for all non-negative integers  $a \leq r-1$  and  $2 \nmid n$ , then the irreducible  $\mathcal{B}_{r,n}(\mathbf{u})$ -modules are indexed by  $(f, \lambda)$  with  $0 \leq f \leq \lfloor n/2 \rfloor$  and  $\lambda$ 's are Kleshchev multipartitions of  $n-2f$ .*
- b) *If  $\omega_a = 0$  for all non-negative integers  $a \leq r-1$  and  $2 \mid n$ , then the irreducible  $\mathcal{B}_{r,n}(\mathbf{u})$ -modules are indexed by  $(f, \lambda)$  with  $0 \leq f < \lfloor n/2 \rfloor$  and  $\lambda$ 's are Kleshchev multipartitions of  $n-2f$ .*

At the end of this section, we recall Goodman's result for  $0 < d < r$  in [14]. In this case,  $d$  is the minimal integer such that  $\{e_1, e_1x_1, \dots, e_1x_1^d\}$  is linear dependent

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<sup>3</sup>In [29],  $\kappa$  is an arbitrary field.

in  $\mathcal{B}_{r,2}(\mathbf{u})$ . Goodman [14] showed that there is a polynomial  $g(x_1) \in \kappa[x_1]$  with  $\deg g(x_1) = d$  such that  $e_1 g(x_1) = 0$  and  $e_1 h(x_1) \neq 0$  in  $\mathcal{B}_{r,2}(\mathbf{u})$  for any polynomial  $h(x_1) \in \kappa[x_1]$  with  $\deg h(x_1) < d$ . Further, since  $e_1 f(x_1) = 0$  in  $\mathcal{B}_{r,2}(\mathbf{u})$ , it is not difficult to see that  $g(x_1) \mid f(x_1)$ . So, write

$$g(x_1) = (x_1 - v_1)(x_1 - v_2) \cdots (x_1 - v_d),$$

where  $\{v_1, v_2, \dots, v_d\} \subset \{u_1, u_2, \dots, u_r\}$  such that  $\mathbf{v}$ -admissible condition holds in  $\mathcal{B}_{d,n}(\mathbf{v})$ . Let  $\langle e_1 \rangle_r$  (resp.  $\langle e_1 \rangle_d$ ) be the two-sided ideal of  $\mathcal{B}_{r,n}(\mathbf{u})$  (resp.  $\mathcal{B}_{d,n}(\mathbf{v})$ ) generated by  $e_1$ .

**Theorem 2.12.** [14, 5.11] *There is an algebraically epimorphism  $\theta : \mathcal{B}_{r,n}(\mathbf{u}) \twoheadrightarrow \mathcal{B}_{d,n}(\mathbf{v})$  such that the restriction of  $\theta$  on  $\langle e_1 \rangle_r$  gives rise to an isomorphism between  $\langle e_1 \rangle_r$  and  $\langle e_1 \rangle_d$ .*

Since  $\mathbf{v}$ -admissible conditions hold in  $\mathcal{B}_{d,n}(\mathbf{v})$ ,  $\langle e_1 \rangle_d$  is cellular with a basis which is given in Theorem 2.9 for  $\mathcal{B}_{d,n}(\mathbf{v})$  with respect to the poset which consists of all pairs  $(f, \lambda) \in \Lambda_{d,n}$  such that  $f \geq 1$ . Via the isomorphism  $\theta$ , Goodman [17] lifted the cellular basis of  $\langle e_1 \rangle_d$  to get the corresponding cellular basis of  $\langle e_1 \rangle_r$ . Using the epimorphism  $\pi : \mathcal{B}_{r,n}(\mathbf{u}) \twoheadrightarrow \mathcal{H}_{r,n}(\mathbf{u})$ , Goodman [14] showed the following result.

**Theorem 2.13.** [14, Theorem 6.4] *Suppose that the  $d$ -semi-admissible condition holds for  $\mathcal{B}_{r,n}(\mathbf{u})$ . Then  $\mathcal{B}_{r,n}(\mathbf{u})$  is (weakly) cellular over the poset*

$$\tilde{\Lambda}_{r,n} = \cup_{1 \leq f \leq \lfloor n/2 \rfloor} \{(f, \lambda) \mid (f, \lambda) \in \Lambda_d^+(n - 2f)\} \cup \{(0, \lambda) \mid \lambda \in \Lambda_r^+(n)\}$$

*in the sense of Definition 2.7. Further,  $\dim_{\kappa} \mathcal{B}_{r,n}(\mathbf{u}) = d^n(2n-1)!! + r^n n! - d^n n!$ .*

We remark that  $(f, \lambda) \leq (\ell, \mu)$  for  $(f, \lambda), (\ell, \mu) \in \tilde{\Lambda}_{r,n}$  if either  $f < \ell$  or  $f = \ell$  and  $\lambda \leq \mu$  where  $\leq$  is the dominance order on  $\Lambda_d^+(n - 2f)$  (resp.  $\Lambda_r^+(n)$ ) provided  $f > 0$  (resp.  $f = 0$ ).

For each  $(f, \lambda) \in \Lambda_{d,n}$  with  $f \geq 1$ , let  $S^{f,\lambda}$  (resp.  $D^{f,\lambda}$ ) be the cell (resp. irreducible) module of  $\mathcal{B}_{d,n}(\mathbf{v})$  with respect the cellular basis in Theorem 2.9. Then  $S^{f,\lambda}$  (resp.  $D^{f,\lambda}$ ) can be considered as the corresponding cell (resp. irreducible) module of  $\mathcal{B}_{r,n}(\mathbf{u})$  with respect to  $(f, \lambda) \in \tilde{\Lambda}_{r,n}$  such that  $f > 0$ . Therefore, we can always assume that  $\mathbf{u}$ -admissible conditions holds when we discuss the irreducible module  $D^{f,\lambda}$  for  $f > 0$ . This is the reason why we add  $\mathbf{u}$ -admissible condition in Theorem 1.2(a)(ii).

### 3. PROOF OF THEOREM 1.2

In this section, we prove Theorem 1.2, which gives the classification of finite dimensional irreducible  $\hat{\mathcal{B}}_n$ -modules over  $\kappa$ .

**Lemma 3.1.** *Suppose  $n > 2$ . If  $\omega_0 \neq 0$ , we define  $e = \omega_0^{-1} e_{n-1}$ . Otherwise, we define  $e = \rho^{-1} e_{n-1} g_{n-2}$ . Then  $e^2 = e$  and  $e \hat{\mathcal{B}}_n e = \hat{\mathcal{B}}_{n-2} e \cong \hat{\mathcal{B}}_{n-2}$  as  $\kappa$ -algebras.*



*Proof.* It follows from Definition 1.1(5)(10) that  $e^2 = e$ . By [18, 3.17, 3.20],  $e_{n-1}\hat{\mathcal{B}}_{n-1}e_{n-1} = e_{n-1}\hat{\mathcal{B}}_{n-2}$  and  $\hat{\mathcal{B}}_ne_{n-1} = \hat{\mathcal{B}}_{n-1}e_{n-1}$ . Therefore,  $e_{n-1}\hat{\mathcal{B}}_ne_{n-1} = e_{n-1}\hat{\mathcal{B}}_{n-2}$ . Now, everything follows since  $g_{n-2}$  is invertible. We remark that the required isomorphism from  $\hat{\mathcal{B}}_{n-2}$  to  $\hat{\mathcal{B}}_{n-2}e$  sending  $x$  to  $xe$  for all  $x \in \hat{\mathcal{B}}_{n-2}$ . One can verify the injectivity of this homomorphism by using the result on the basis of  $\hat{\mathcal{B}}_n$  in [18].  $\square$

Let  $\hat{\mathcal{B}}_n\text{-mod}$  be the category of finite dimensional right  $\hat{\mathcal{B}}_n$ -modules over  $\kappa$ . By Lemma 3.1, we have the functor  $\mathfrak{F} : \hat{\mathcal{B}}_n\text{-mod} \rightarrow \hat{\mathcal{B}}_{n-2}\text{-mod}$  such that

$$(3.2) \quad \mathfrak{F}(M) = Me$$

for any object  $M \in \hat{\mathcal{B}}_n\text{-mod}$ . Further, if  $M$  is a  $\mathcal{B}_{r,n}(\mathbf{u})$ -module and if there is an epimorphism  $\phi : \hat{\mathcal{B}}_n \twoheadrightarrow \mathcal{B}_{r,n}(\mathbf{u})$ , then  $\mathfrak{F}(M)$  is the same as  $\mathcal{F}(M)$  where  $\mathcal{F}$  is the exact functor from  $\mathcal{B}_{r,n}(\mathbf{u})\text{-mod}$  to  $\mathcal{B}_{r,n-2}(\mathbf{u})\text{-module}$ . However, by Theorem 2.13 and the statements below Theorem 2.13, we can always assume the  $\mathbf{u}$ -admissible condition holds when we discuss  $S^{f,\lambda}$  and  $D^{f,\lambda}$  for  $f > 0$ . In this case, by [28, Sect. 5], we have

$$(3.3) \quad \mathcal{F}(S^{f,\lambda}) = S^{f-1,\lambda} \text{ and } \mathcal{F}(D^{f,\lambda}) = D^{f-1,\lambda}.$$

Note that  $D^{f,\lambda} \neq 0$  if and only if  $D^\lambda \neq 0$  (see Proposition 2.10). Further,

$$\mathcal{F}(S^{0,\lambda}) = \mathcal{F}(D^{0,\lambda}) = 0$$

no matter whether the  $\mathbf{u}$ -admissible condition holds for  $\mathcal{B}_{r,n}(\mathbf{u})$ .

**Lemma 3.4.** *Suppose  $(f, \lambda) \in \Lambda_{r,n}$  (resp.  $(\ell, \mu) \in \Lambda_{s,n}$ ) such that  $D^{f,\lambda} \neq 0$  (resp.  $D^{\ell,\mu} \neq 0$ ) as  $\mathcal{B}_{r,n}(\mathbf{u})$ -module (resp.  $\mathcal{B}_{s,n}(\mathbf{v})$ -module). If both  $\mathcal{B}_{r,n}(\mathbf{u})$  and  $\mathcal{B}_{s,n}(\mathbf{v})$  are images of  $\hat{\mathcal{B}}_n$  such that  $D^{f,\lambda} \cong D^{\ell,\mu}$  as  $\hat{\mathcal{B}}_n$ -modules, then  $f = \ell$  and  $D^\lambda \cong D^\mu$  as  $\hat{\mathcal{H}}_{n-2f}$ -modules.*

*Proof.* If  $f \neq \ell$ , we can assume that  $f \geq \ell + 1$  without loss of any generality. By Theorem 2.13, we can always assume that  $\mathbf{u}$ -admissible (resp.  $\mathbf{v}$ -admissible) condition holds (resp. if  $\ell \neq 0$ ).

Applying the functor  $\mathfrak{F}$  on both  $D^{f,\lambda}$  and  $D^{\ell,\mu}$  repeatedly yields  $D^{f-\ell-1,\lambda} = 0$  as  $\hat{\mathcal{B}}_{n-2\ell-2}$ -modules. By Proposition 2.10,  $D^{f-\ell-1,\lambda} \neq 0$ , a contradiction. So,  $f = \ell$ . Applying the functor  $\mathfrak{F}$  on both  $D^{f,\lambda}$  and  $D^{f,\mu}$  yields  $D^{0,\lambda} \cong D^{0,\mu}$  as  $\hat{\mathcal{B}}_{n-2f}$ -modules. In other words,  $D^{0,\lambda} \cong D^{0,\mu}$  as  $\hat{\mathcal{H}}_{n-2f}$ -modules. Note that  $D^{0,\lambda}$  (resp.  $D^{0,\mu}$ ) can be identified with  $D^\lambda$  (resp.  $D^\mu$ ) as  $\mathcal{H}_{r,n-2f}(\mathbf{u})$  (resp.  $\mathcal{H}_{s,n-2f}(\mathbf{v})$ )-module. Now, everything follows.  $\square$

**Lemma 3.5.** *Suppose  $(f, \lambda) \in \Lambda_{r,n}$  (resp.  $(f, \mu) \in \Lambda_{s,n}$ ) such that  $D^{f,\lambda} \neq 0$  (resp.  $D^{f,\mu} \neq 0$ ) as  $\mathcal{B}_{r,n}(\mathbf{u})$ -module (resp.  $\mathcal{B}_{s,n}(\mathbf{v})$ -module). If both  $\mathcal{B}_{r,n}(\mathbf{u})$  and  $\mathcal{B}_{s,n}(\mathbf{v})$  are images of  $\hat{\mathcal{B}}_n$  and if  $D^\lambda \cong D^\mu$  as  $\hat{\mathcal{H}}_{n-2f}$ -modules, then  $D^{f,\lambda} \cong D^{f,\mu}$  as  $\hat{\mathcal{B}}_n$ -modules.*

*Proof.* First, we can assume  $f \neq 0$ . Otherwise, there is nothing to be proved. By assumption,  $D^{f,\lambda}$  (resp.  $D^{f,\mu}$ ) is the irreducible  $\mathcal{B}_{r,n}(\mathbf{u})$ -module (resp.  $\mathcal{B}_{s,n}(\mathbf{v})$ -module) with respect to  $(f, \lambda) \in \Lambda_{r,n}$  (resp.  $(f, \mu) \in \Lambda_{s,n}$ ). Suppose

$$\mathcal{B}_{r,n} = \hat{\mathcal{B}}_n/I \text{ and } \mathcal{B}_{s,n} = \hat{\mathcal{B}}_n/J,$$

where  $I$  (resp.  $J$ ) is the two-sided ideal of  $\hat{\mathcal{B}}_n$  generated by  $f(x_1)$  (resp.  $g(x_1)$ ) and

$$\begin{aligned} f(x_1) &= (x_1 - u_1)(x_1 - u_2) \cdots (x_1 - u_r), \\ g(x_1) &= (x_1 - v_1)(x_1 - v_2) \cdots (x_1 - v_s). \end{aligned}$$

Let  $h(x) = [f(x_1), g(x_1)]$  be the least common multiple of  $f(x_1)$  and  $g(x_1)$ . Let  $\mathcal{B}_{t,n} = \hat{\mathcal{B}}_n/K$  where  $K$  is the two-sided ideal of  $\hat{\mathcal{B}}_n$  generated by  $h(x_1)$ . Then there are two algebraical epimorphisms:

$$\phi : \mathcal{B}_{t,n} \twoheadrightarrow \mathcal{B}_{r,n}(\mathbf{u}), \text{ and } \psi : \mathcal{B}_{t,n} \twoheadrightarrow \mathcal{B}_{s,n}(\mathbf{v})$$

such that  $\phi$  (resp.  $\psi$ ) sends generators  $e_i, g_i, x_1 \in \mathcal{B}_{t,n}$  to the corresponding generators  $e_i, g_i, x_1$  in  $\mathcal{B}_{r,n}(\mathbf{u})$  (resp.  $\mathcal{B}_{s,n}(\mathbf{v})$ ),  $1 \leq i \leq n-1$ .

In particular, by Lemma 3.4 and Theorem 2.13, the irreducible  $\mathcal{B}_{r,n}(\mathbf{u})$ -module (resp.  $\mathcal{B}_{s,n}(\mathbf{v})$ -module)  $D^{f,\lambda}$  (resp.  $D^{f,\mu}$ ) has to be the irreducible  $\mathcal{B}_{t,n}$ -module  $D^{f,\alpha}$  (resp.  $D^{f,\beta}$ ) for some multi-partition  $\alpha$  (resp.  $\beta$ ) such that  $D^\alpha \cong D^\lambda$  and  $D^\beta \cong D^\mu$  as  $\hat{\mathcal{H}}_{n-2f}$ -modules. Further, by the arguments below Theorem 2.13 and the results on the representation theory for cyclotomic BMW algebras  $\mathcal{B}_{t,n}$ , we have that both  $D^\alpha$  and  $D^\beta$  are irreducible modules for the same Ariki-Koike algebra. Since we are assuming that  $D^\lambda \cong D^\mu$  as  $\hat{\mathcal{H}}_{n-2f}$ -modules, we have  $D^\alpha \cong D^\beta$ , forcing  $\alpha = \beta$ . So,  $D^{f,\lambda} \cong D^{f,\alpha} \cong D^{f,\mu}$  as  $\hat{\mathcal{B}}_n$ -modules, proving the result.  $\square$

**Proof of Theorem 1.2:** Let  $D^\lambda$  be the irreducible modules for  $\mathcal{H}_{r,n}(\mathbf{u})$ . In [1], Ariki has proved that  $D^\lambda \neq 0$  if and only if  $\lambda$  is Kleshchev in the sense of Definition in [4, p605]. On the other hand, simple modules for affine Hecke algebra  $\hat{\mathcal{H}}_n$  can be labeled by aperiodic multisegments [4]. In [3], Ariki, Jacon and Lecouvey set up the explicit relationship between the set of Kleshchev multipartitions and the set of aperiodic multisegments in [3, Theorem 6.2]. In other words, if  $D^\lambda$  and  $D^\mu$  are irreducible modules for different Ariki-Koike algebras with respect to the Kleshchev multi-partitions  $\lambda$  and  $\mu$ , then  $D^\lambda \cong D^\mu$  as irreducible modules for extended affine Hecke algebra if and only if the images of  $\lambda$  and  $\mu$  with respect to (different) map in [3, Theorem 6.2] are the same aperiodic multi-segment with length  $n - 2f$ . Further, when  $f > 0$ , we have to assume the  $\mathbf{u}$ -admissible condition hold. However, when  $f = 0$ , we do not need this assumption. Now, everything follows from Lemmas 3.4-3.5 and 2.6.  $\square$

We close the paper by giving the following remark.

**Remark 3.6.** We can classify the finite dimensional irreducible modules for affine Wenzl algebra over an algebraically closed field  $\kappa$ . In this case, we have to use the results for degenerate affine Hecke algebra of type  $A_{n-1}$  instead of those for  $\hat{\mathcal{H}}_n$ . We leave the details to the reader.

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